Information market games

Muto, S.; Potters, J.A.M.; Tijs, S.H.

Published in:
International Journal of Game Theory

Publication date:
1989

International Journal of Game Theory (1989) 18: 209–226

Information Market Games

By S. Muto[1], J. Potters[2] and S. Tijs[2]

Abstract: In this paper information markets with perfect patent protection and only one initial owner of the information are studied by means of cooperative game theory. To each information market of this type a cooperative game with sidepayments is constructed. These cooperative games are called information (market) games. The set of all information games with fixed player set is a cone in the set of all cooperative games with the same player set. Necessary and sufficient conditions are given in order that a cooperative game is an information game. The core of this kind of games is not empty and is also the minimal subsolution of the game. The core is the image of an (n-1)-dimensional hypercube under an affine transformation, (= hyperparallellopiped), the nucleolus and τ-value coincide with the center of the core. The Shapley value is computed and may lie inside or outside the core. The Shapley value coincides with the nucleolus and the τ-value if and only if the information game is convex. In this case the core is also a stable set.

1 Introduction

Recently, many papers appeared dealing with the issue of intentional sharing of concealible (i.e. easily kept secret if not disseminated) information. Assuming perfect patent protection, Gallini (1984), Gallini/Winter (1984), Katz/Shapiro (1984 a, 1984 b, 1984 c), Kamien/Tauman (1984, 1986) and Kamien/Tauman/Zang (1985) examined the incentives to disseminate by licensing a cost reducing process innovation in duopolistic or oligopolistic markets. Muto (1987) examined the dissemination of a process innovation in the case of imperfect patent protection, in which relicensing by licensees may take place. In more general contexts, Güth (1984) and Muto (1986) showed that the possessor of a replicatable information has the incentive of selling it, even if resales of its replicas by the buyers are freely allowed. All of these analyses were done in terms of noncooperative game theory.

Concerning cooperative behaviour of traders, only a few works are observed. Katz/Tauman (1982) studied cooperation among a patent holder and producers in an oligopolistic market; and Nakayama (1985) studied cooperation between traders in an information market with imperfect patent protection.

1 Shigeo Muto, Faculty of Economics, Tohoku University, Kawauchi, Sendai 980, Japan.

2 Jos Potters and Stef Tijs, Department of Mathematics, Catholic University Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands.

0020–7276/89/2/209–226 $2.50

The aim of this paper is to analyze the cooperative behaviour of economic agents (firms), faced with the introduction of a *new* technology, indispensible for the manufacturing of a *new* product. We assume that the (consumer) market of this new product is divided into parts (submarkets) according to the group of firms which have the right and possibility to enter this market. For each submarket one knows the maximal profit which can be achieved by producing and selling the new commodity.

The information is initially only possessed by one of the firms, the patent holder. Since we assume *perfect patent protection* the initial owner of the information has the possibility to monopolize every submarket he has entrance to. Each of the other firms is endowed with the potential ability of manufacturing the new commodity and can attain the maximal profit of each submarket provided the firm has the information, has entrance to the submarket and meets no competition of other informed firms. By sharing the information with other firms (licensing), the patent holder can indirectly also gain profit from submarkets he has no access to. But patent protection forbids further dissemination of the information without the permission of the patent holder. Or, more formally we define:

An *information market* (with one initially informed player) consists of the following data

$$< N, \{1\}, < M_T, r_T >_{T \subset N,\, T \neq \emptyset} >$$

where $N = \{1,...,n\}$ is the set of firms and firm 1 is the initially informed firm, the patent holder. Further, $\{M_T\}_{T \subset N,\, T \neq \emptyset}$ is the partioning of the consumer market into submarkets. Here M_T is the submarket to which the firms of T has access to and no other firms. Moreover, $r_T \in \mathbb{R}_+$ is the maximal profit obtainable from the submarket M_T.

Remark: Submarkets may be empty or may have profit $r_T = 0$.

From these data the initially informed firm has to decide whether and, if so, for what price and to which firms he will sell the license rights; the other firms have to decide for what price they will purchase these rights.

The remainder of the paper is organized as follows. In section 2, the market situation will be formulated as a cooperative game with side payments. We shall call the games in this way obtained, *information (market) games.* Also in section 2 we characterize in two ways the information games inside the set of all cooperative games. Solution concepts as the core, stable sets and subsolutions, the Shapley value, the nucleolus and the τ-value are studied in sections 3 and 4. The paper closes in section 5 with a short summary of the results.

2 Information Games

Let us suppose that an information market is given:

$$< N, \{1\}, < M_T, r_T >_{T \subset N, T \neq \emptyset} >$$

We shall define a *cooperative game with sidepayments* generated by this information market model.

A cooperative game (with sidepayments) with player set N is a map (*the characteristic function*) $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Here $v(S)$ is the *value* or *worth* of coalition S. The collection of all cooperative games with player set N we denote by G^N. Note that G^N is a linear subspace of $\mathbb{R}^{2^N}$ of dimension 2^n-1.

In the following way an information market gives rise to a cooperative game: N is the set of firms and $v(S)$ is the profit which the coalition S can gain *without the help of any firm outside of coalition S*. For coalitions S, not containing the initially informed firm 1, this means that $v(S) = 0$. If firm 1 is in the coalition S, then the firms of S can produce and sell for each submarket to which at least one firm of S has access. This means that they can achieve the profit

$$v(S) = \sum_{T : T \cap S \neq \emptyset} r_T.$$

We shall call a cooperative game which arises in this way from an information market an *information (market) game*. The set of all information games with player set N is denoted by IG^N.

The remainder of this section will be devoted to characterize the set IG^N and to characterize the elements of IG^N. First we recall some definitions.

A cooperative game v is called *monotonic* if
$v(S) \leq v(T)$ for all pairs of coalitions S and T with $S \subset T$.
A cooperative game v is called *superadditive* if
$v(S)+v(T) \leq v(S \cup T)$ for all disjoint coalitions S and T.
The dual game v^* corresponding to a game $v \in G^N$ is the game v^* with
$v^*(S) = v(N)-v(N \setminus S)$ for all $S \subset N$.

Information games are monotonic since $r_T \geq 0$ for all $T \subset N$. Information games are also superadditive because disjointness of S and T implies that at least one coalition does not contain 1 and hence has value zero and then superadditivity follows from monotonicity.

A *simple game* is a cooperative game v with the following properties:

(i) $v(S) \in \{0,1\}$ for all coalitions S
(ii) v is monotonic
(iii) $v(N) = 1$.

In a simple game the coalitions S with $v(S) = 1$ are called *winning coalitions* and winning coalitions S with the property that every proper subcoalition T of S has value $v(T) = 0$ are called *minimal winning coalitions.*

Simple games are uniquely determined by the set of minimal winning coalitions:

$$v(S) = 1 \Leftrightarrow S \text{ contains at least one minimal winning coalition.}$$

Notice that of two different minimal winning coalitions S_1 and S_2 no one can contain the other.

For $T \subset N \setminus \{1\}$ let $u_{T,1}$ be the simple game with minimal winning coalitions T and $\{1\}$. Let u_1 be the simple game with minimal winning coalition $\{1\}$ only.

Now we find the following characterization of IG^N inside G^N:

Theorem 2.1: IG^N is the cone generated over $\mathbb{R}_+$ by u_1 and $\{u^*_{T,1} \mid T \subset N \setminus \{1\}, T \neq \emptyset\}$.

Proof: (i) Suppose, v is an information game generated by the market data $\{r_T\}_{T \subset N, T \neq 0}$. Let $r := \sum_{T1 : \epsilon T} r_T$. We show that $v = ru_1 + \sum_{T \subset N \setminus \{1\}} r_T u^*_{T,1}$.

In fact, if $1 \notin S$, then $v(S) = 0$, $u_1(S) = 0$ and $u^*_{T,1}(S) = u_{T,1}(N) - u_{T,1}(N \setminus S) = 0$.

If $1 \in S$, then $u_1(S) = 1$ and $u^*_{T,1}(S) = 1 \Leftrightarrow T \not\subset N \setminus S \rightarrow T \cap S \neq 0$.
So $(r u_1 + \Sigma r_T u^*_{T,1})(S) = r + \sum_{T : T \cap S \neq \emptyset, T \subset N \setminus \{1\}} r_T = \sum_{T : T \cap S \neq \emptyset} r_T = v(S)$.

(ii) Conversely, let $v = \lambda u_1 + \sum_{T : T \subset N \setminus \{1\}} \lambda_T u^*_{T,1}$ with $\lambda_T \geq 0$ for all $T \cup N \setminus \{1\}$, $T \neq \emptyset$.

Then this game is generated by every information market with

$$r_T = \lambda_T \text{ if } T \subset N \setminus \{1\} \text{ and } \sum_{T : 1 \epsilon T} r_T = \lambda \qquad \square$$

The following theorem characterizes information games as games with certain specific properties. In this characterization we use the idea of *derivatives of a cooperative game* (cf. Shapley (1971)).

Let v be a cooperative game and $U \subset N$. Then the U-derivative of v is the map

$$\Delta_U(v) : 2^{N \setminus U} \rightarrow \mathbb{R}$$

defined by:

$$\Delta_U(v)(S) = \sum_{T : T \cup U} (-1)^{|U| - |T|} v(S \cup T) \text{ for all } S \subset N \setminus U$$

where $|U|$ (and $|T|$) are the cardinalities of U (resp. T).

For example: $\Delta_\phi(v) = v$;

$\Delta_{\{i\}}(v)(S) = v(S \cup \{i\}) - v(S)$, the marginal contribution of i to coalition $S \cup \{i\}$;

$\Delta_{\{i,j\}}(v)(S) = v(S \cup \{i,j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)$.

In particular: v is monotonic iff $\Delta_U(v) \geq 0$ for all coalitions U with $|U| = 1$ and v is convex iff $\Delta_U(v) \geq 0$ for all U with $|U| = 2$.

In the next theorem we characterize information games with the aid of derivatives.

Theorem 2.2: A cooperative game $v \in G^N$ is an information game if and only if

(i) $v(\{1\}) \geq 0$

(ii) $v(S) = 0$ for all $S \subset N \setminus \{1\}$

(iii) $(-1)^{|U|-1} \Delta_U(v)(S) \geq 0$ for all $U \subset N$, $U \neq \emptyset$ and for all $S \subset N \setminus U$, containing 1.

In order to prove this theorem we first show the following

Lemma 2.3: $\{u^*_{T,1} \mid T \subset N \setminus \{1\}, T \neq \emptyset\}$ is a basis of the linear subspace of G^N consisting of all cooperative games v with $v(S) = 0$ if $|S| = 1$ or $1 \in S$.

Proof: The dimension of $\{v \in G^N \mid v(S) = 0$ if $|S| = 1$ or $1 \notin S\}$ is $2^{n-1} - 1$, just equal to the number of $T \subset N \setminus \{1\}$, $T \neq \emptyset$.

So, we have to show only the linear independence of $\{u^*_{T,1} \mid T \subset N \setminus \{1\}, T \neq \emptyset\}$. In order to prove this fact we compute $\Delta_U(u^*_{T,1})(S)$ with $1 \in S \subset N$, $U \subset N \setminus S$, $U \neq \emptyset$ and $T \subset N \setminus \{1\}$, $T \neq \emptyset$.

Note that from Newton's binomial formula follows $\sum_{P: K \subset P \subset L} (-1)^{|P|} = 0$ unless $K = L$. (2.1)

$$\Delta_U(u^*_{T,1})(S) = \sum_{V: V \subset U} (-1)^{|U|-|V|} u^*_{T,1}(S \cup V) =$$

$$\sum_{V: V \subset U;\ T \cap (S \cup V) \neq \emptyset} (-1)^{|U|-|V|}$$

In case $S \cap T \neq \emptyset$ we may skip the condition $T \cap (S \cup V) \neq \emptyset$ and we find

$$\sum_{V: V \subset U} (-1)^{|U|-|V|} = 0 \text{ by (2.1) since } U \neq \emptyset.$$

Now we suppose $T \subset N\backslash S$ and find:

$$\Delta_U(u^*_{T,1}) = \sum_{V:\, V\subset U;\; V\cap T\neq\emptyset} (-1)^{|U|-|V|} = \sum_{V:\, V\subset U} x\,(-1)^{|U|-|V|}$$

$$- \sum_{V:\, V\subset U;\; V\cap T=\emptyset} (-1)^{|U|-|V|} = (-1)^{|U|-1} \sum_{V:\, V\subset U\backslash T} (-1)^{|V|} = 0 \text{ unless } U\backslash T=$$

$\emptyset$; in that case $\Delta_U(u^*_{T,1})\,(S) = (-1)^{|U|-1}$.

So $\Delta_U(u^*_{T,1})\,(S) = (-1)^{|U|-1}\,\delta\,(U \subset T \subset N\backslash S)$ where

$$\delta\,(U \subset T \subset N\backslash S) = \begin{cases} 1 \text{ if } U \subset T \subset N\backslash S \\ 0 \text{ otherwise.} \end{cases}$$

Now suppose $\Sigma\, \lambda_T\, u^*_{T,1} = 0$ with $\lambda_T \in \mathbb{R}$. For $U \subset N\backslash\{1\}$ and $U \neq \emptyset$ we find

$$0 = \Delta_U(\Sigma\, \lambda_T\, u^*_{T,1})\,(N\backslash U) = \sum_{T\subset N\backslash\{1\};\; T\neq\emptyset} \lambda_T\, \Delta_U(u^*_{T,1})\,(N\backslash U)$$

$$= \sum_T \lambda_T\,(-1)^{|U|-1}\,\delta\,(U \subset T \subset U) = (-1)^{|U|-1}\,\lambda_U.$$

The linear independency follows. □

Proof of Theorem 2.2: Suppose $v \in IG^N$. Then (i) and (ii) are trivial. From theorem 2.1 follows

$$v = r u_1 + \sum_{T\subset N\backslash\{1\};\; T\neq\emptyset} r_T\, u^*_{T,1}\,.$$

From the linearity of Δ_U and the fact that $\Delta_U(u_1) = 0$ if $U \neq \emptyset$ and $1 \notin U$ we find

$$(-1)^{|U|-1}\,\Delta_U(v)\,(S) = (-1)^{|U|-1} \sum_{T:\, T\subset N\backslash\{1\};\; T\neq\emptyset} r_T\,\delta\,(U\subset T\subset N\backslash S)\,(-1)^{|U|-1}$$

$$= \sum_{T:\, U\subset T\subset N\backslash S} r_T \geq 0 \text{ if } U \neq \emptyset \text{ proving (iii).}$$

Conversely, let v be a cooperative game satisfying (i), (ii) and (iii). Then $v = v(\{1\})u_1 + \tilde{v}$ and $\tilde{v}(S) = 0$ if $|S| = 1$ or $1 \notin S$.
By lemma 2.3

$$\tilde{v} = \sum_{T\subset N\backslash\{1\}} \lambda_T\, u^*_{T,1}\,.$$

Compute as before $(-1)^{|U|-1}\,\Delta_U(\tilde{v})\,(N\backslash U)$ for $U \subset N\backslash\{1\}$ and $U \neq \emptyset$ and we find that $\lambda_U \geq 0$ for all $U \subset N\backslash\{1\}$ and $U \neq \emptyset$ by property (iii). So

$$v = v(\{1\})\,u_1 + \sum_{T:T\subset N\backslash\{1\}} \lambda_T\, u^*_{T,1} \quad \text{with } \lambda_T \geq 0 \text{ for all } T \subset N\backslash\{1\}.$$

Hence, v is an information game by theorem 2.1 □

Remarks:

(i) The zero-normalization of an information game generated by the market data $\{r_T\}_{T\subset N,\ T\neq\emptyset}$ is the information game generated by $\{\tilde{r}_T\}_{T\subset N,\ T\neq\emptyset}$ such that $\tilde{r}_T = r_T$ if $1 \notin T$ and $\tilde{r}_T = 0$ if $1 \in T$.

(ii) The market data r_T with $T \subset N\backslash\{1\}$ can be rediscovered from the information game they generate, namely by

$$r_T = (-1)^{|T|-1}\,\Delta_T(v)\,(N\backslash T).$$

(iii) As far as it concerns coalitions containing 1, v is a concave game:

$$v(S)+v(T) \geq v(S \cup T)+v(S \cap T) \text{ if } 1 \in S \cap T.$$

3 The Core, Stable Sets and Subsolution of Information Games

The set of *imputations* $A(v)$ of a cooperative game v is the set of all vectors $x \in \mathbb{R}^N$ with

$$x_i \geq v(\{i\}) \text{ for all } i \in N \text{ and } x(N) = \sum_{i\in N} x_i = v(N).$$

For superadditive games v the core $C(v)$ is the set of all imputations $x \in A(v)$ with

$$x(S) = \sum_{i\in S} x_i \geq v(S) \text{ for all coalitions } S \subset N.$$

The following theorem gives a description of the core of an information game saying that core elements give the uninformed players at most the profit of the market they govern alone.

Theorem 3.1: The core of an information game v generated by the market data

$\{r_T\}_{T \subset N,\, T \neq \emptyset}$ is equal to the set

$\{x \in \mathbb{R}^N \mid x(N) = v(N) : 0 \leq x_i \leq r_i$ for all $i \in N \backslash \{1\}\}$.

Proof: Every core element x satisfies the inequalities:

$x_i = x(N) - x(N \backslash \{i\}) \leq v(N) - v(N \backslash \{i\})$ and $x_i \geq v(\{i\}) = 0$ for ali $i \in \{2,\ldots,n\}$.

But $v(N) - (N \backslash \{i\}) = \sum_T r_T - \sum_{T \cap (N \backslash \{i\}) \neq \emptyset} r_T = r_i$ for $i \in \{2,\ldots,n\}$.

Hence $0 \leq x_i \leq r_i$ for $i \in \{2,\ldots,n\}$.

If, conversely, $0 \leq x_i \leq r_i$ for $i \in \{2,\ldots,n\}$ and $x(N) = v(N)$, then

$x(S) \geq 0 = v(S)$ if $1 \notin S$

and, if $1 \in S$, $x(S) = x(N) - x(N \backslash S) \geq v(N) - r(N \backslash S) \geq \sum_{T \cap S \neq \emptyset} r_T = v(S)$ □.

Example: Suppose $N = \{1,2,3\}$; $r_{\{1,3\}} = 200$, $r_{\{2\}} = 500$ and $r_T = 0$ for all other $T \subset N$. Then $v(1) = v(1,3) = 200$, $v(1,2) = v(1,2,3) = 700$ and $v(S) = 0$ for all other coalitions S.

The core is $C(v) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 = 200+p,\ x_2 = 500-p,\ x_3 = 0,\ 0 \leq p \leq 500\}$.

The core of information games appears to have more stability properties than one may expect in general: the core is also a *subsolution* of the game, a concept introduced by Roth (1976).

Let $A(v)$ be the set of imputations of a cooperative game v. For $x, y \in A(v)$ we say: *x dominates y with respect to coalition S* (notation: $x \operatorname{dom}_S y$) if

(i) $x_i > y_i$ for all $i \in S$
(ii) $x(S) \leq v(S)$.

We say: x *dominates* y if there exists a coalition S such that $x \text{ dom}_S\, y$. (Notation x dom y.)

If $K \subset A(v)$, then $\quad \text{dom}\, K = \{y \in A(v) \mid \exists\, x \in K : x \text{ dom } y\}$.

A subset $K \subset A(v)$ is called a *stable set* or a *von Neumann/Morgenstern solution* if

(i) $K \cap \text{dom}\, K = \emptyset$ (internal stability)
(ii) $K \cup \text{dom}\, K = A(v)$ (external stability).

(cf. von Neumann/Morgenstern (1953))

A subset $L \subset A(v)$ is called a *subsolution* (cf. Roth (1976)) if

(i) $L \cap \text{dom}\, L = \emptyset$
(ii) if $x \in L$ and y dom x, then $y \in \text{dom}\, L$
(iii) if $x \in A(v)$ and $x \notin L \cup \text{dom}\, L$, then there is an imputation $y \notin L \cup \text{dom}\, L$ with y dom x.

In the next proposition we collect some well-known properties of the dominance relation.

Proposition 3.2: For all cooperative games v the following properties hold:

(i) if $x, y \in A(v)$ and $x \text{ dom}_S\, y$, then $|S| \geq 2$.
(ii) $C(v) \cap \text{dom}\, A(v) = \emptyset$ i.e. core elements are undominated.
(iii) Every subsolution contains the core.
(iv) The core $C(v)$ is a subsolution if and only if it satisfies property (iii) in the definition of subsolution. □

For *information games* we have:

Theorem 3.3: If $v \in IG^N$, then $C(v)$ is a subsolution.

Proof: In view of (iv) of proposition 3.2 we only have to check property (iii). Let $x \in A(v)$ be an imputation outside the core and not dominated by core elements. We have to find an imputation y with the same properties such that y dom x.

Since $x \notin C(v)$, there exists at least one index $i_0 \in \{2,\ldots,n\}$ such that $x_{i_0} > r_{i_0}$.

Choose $\delta_j > 0$ for all $j \neq i_0$ and $\epsilon > 0$ such that

(a) $x_{i_0} - \epsilon > r_{i_0}$

(b) $\sum_{j \neq i_0} \delta_j = \epsilon.$

Define $y_j = x_j + \delta_j$ for $j \neq i_0$ and $y_{i_0} = x_{i_0} - \epsilon$. Then $y \in A(v)$. Furthermore $y_j > x_j$ for $j \neq i_0$ and by (a)

$$y(N\backslash\{i_0\}) = v(N) - y_{i_0} = v(N) - (x_{i_0} - \epsilon) \leq v(N) - r_{i_0} = v(N\backslash\{i_0\}).$$

Hence $y \text{ dom }_{N\backslash\{i_0\}} x$.

$y \notin C(v)$ since $y_{i_0} > r_{i_0}$ by (a) and theorem 3.1.

Suppose $y \in \text{dom } C(v)$ i.e. there is an element $z \in C(v)$ and a coalition S such that $z \text{ dom}_S\, y$.

Because $y_i < z_i \leq r_i$ for all $i \in S\backslash\{1\}$, we have $i_0 \notin S$ and

$z_i > y_i > x_i$ for all $i \in S$.

Then $z(S) \leq v(S)$ implies $z \text{ dom}_S\, x$ and $x \in \text{dom } C(v)$. But x is not dominated by core elements. So $y \notin C(v) \cup \text{dom } C(V)$. □

The following theorem characterizes the information games (markets) for which the core is the unique stable set:

Theorem 3.4: For an information game v generated by market data $\{r_T\}_{T \subset N,\, T \neq \emptyset}$, the following statements are equivalent:

(i) v is a convex game
(ii) the core is a stable set
(iii) $r_T = 0$ for all coalitions $T \subset N\backslash\{1\}$ with $|T| \geq 2$.

Proof: (i) → (ii) is a well-known result of Shapley (1971).
(ii) → (iii) If $r_T > 0$ for some coalition $T \subset N\backslash\{1\}$ with $|T| \geq 2$, then we can find an imputation $y \in A(v)$ not dominated by any core element. Take $y_i = r_i + (n-1)^{-1} r_T$ for $i \in \{2,\ldots,n\}$ and $y_1 = v(N) - \sum_{j \geq 2} y_j$. Then, for every core element x, $y_i > x_i$ for $i \in \{2,\ldots,n\}$ and further $y_1 \geq v(\{1\})$. So, $y \notin C(v)$ and y is not dominated by core elements. Thus $C(v)$ is not a stable set.

(iii) → (i) Let $S,T \subset N$. If $1 \notin S \cap T$ then $v(S) = 0$ or $v(T) = 0$ and $v(S \cap T) = 0$ and

$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ is true by the monotonicity of the game.

If $1 \in S \cap T$ then

$$v(S \cup T) + v(S \cap T) - v(S) - v(T) = \Sigma \{r_U \mid U \cap (S \cup T) \neq \emptyset\}$$
$$+ \Sigma \{r_U \mid U \cap (S \cap T) \neq \emptyset\} - \Sigma \{r_U \mid U \cap S \neq \emptyset\} - \Sigma \{r_U \mid U \cap T \neq \emptyset\} =$$
$$- \Sigma \{r_U \mid U \cap S \cap T = \emptyset, U \cap S \neq \emptyset, U \cap T \neq \emptyset\} \tag{3.1}$$

Suppose that $U \cap S \cap T = \emptyset$, $U \cap S \neq \emptyset$ and $U \cap T \neq \emptyset$. Then $1 \notin U$.

If $|U| = 1$ then $U \cap S \neq \emptyset$ and $U \cap T \neq \emptyset$ imply $U \subset S \cap T$ and $U \cap S \cap T \neq \emptyset$. Hence we find $U \cap S \cap T = 0$, $U \cap S \neq \emptyset$ and $U \cap T \neq \emptyset$ imply $U \subset N\backslash\{1\}$ and $|U| \geq 2$. Hence $r_U = 0$ and the last term of (3.1) is zero.

Therefore $v(S \cup T) + v(S \cap T) = v(S) + v(T)$ if $1 \in S \cap T$ and v is a convex game. □

Remark: In section 4 we shall prove that an information game v is convex iff the Shapley value coincides with τ-value and the nucleolus the game.

In Muto/Potters/Tijs (1987) there will be a discussion on the stable sets of more general information games with only one informed player. In particular there will be a desription of a symmetric stable set of information games with four players and symmetry between the uninformed players i.e. for $T \subset N\backslash\{1\}$ r_T is only dependent on $|T|$.

4 The Shapley Value, Nucleolus and τ-Value of Information Games

In this section we will examine how the three one-point solution concepts mentioned in the title behave in the set of information games. It will turn out that the nucleolus and τ-value coincide for this type of games. As a corollary we find that not only the Shapley value but also the nucleolus and the τ-value are additive on IG^N.

We will start with the τ-value (Tijs (1981), Tijs (1986)).

Theorem 4.1: The τ-value of information games lies in the center of the core i.e.

$$\tau(v) = v(N)e_1 + \sum_{½ i \geq 2} r_i \, (e_i - e_1)$$

where $\{e_i\}_{i \in N}$ is the standard basis of IR^N.

Proof: By definition the τ-value is the element of $A(v)$ which lies on the line segment $[m(v), M(v)]$ where $M(v)$ and $m(v)$ are the n-dimensional vectors with

$$M_i(v) := v(N) - v(N\backslash\{i\}) \text{ and } m_i(v) = \max_{S:i\in S} [v(S) - \sum_{j\in S\backslash\{i\}} M_j(v)]$$

for each $i \in N$.

We compute $M(v)$ and $m(v)$: $M_1(v) = v(N) - v(N\backslash\{1\}) = v(N)$ and for $i \geq 2$: $M_i(v) = v(N) - v(N\backslash\{i\}) = r_i$.
If $i \geq 2$ and $1 \in S$ then $v(S) - M(v)\,(S\backslash\{i\}) \leq v(S) - M_1(v) = v(S) - v(N) \leq 0$.
If $i \geq 2$ and $1 \notin S$ then $v(S) = 0$ and $-M(v)\,(S\backslash\{i\}) \leq 0$.
For $S = \{i\}$ we find $v(S) - M(v)\,(S\backslash\{i\}) = v(\{i\}) = 0$.
Hence $m_i(v) = 0$ for $i \in \{2,...,n\}$.
If $1 \in S$, then

$$v(S) - M(v)\,(S\backslash\{1\}) = \sum_{T:T\cap S\neq 0} r_T - r(S\backslash\{1\}).$$

This number does not decrease if we increase the coalitions S. So

$$m_1(v) = v(N) - M(v)\,(N\backslash\{1\}) = v(N) - r\,(N\backslash\{1\}).$$

We may conclude that the τ-value $\tau(v)$ equals $(\frac{1}{2})\,(M(v) + m(v)) = (v(N) - \frac{1}{2}r(N\backslash\{1\}), \frac{1}{2}r_2,..., \frac{1}{2}r_n)$ □.

Note that the τ-value assigns to a player $i \neq 1$ only half of the profit of the submarket which he governs alone, the other half of these profits and all the profits gained in shared submarkets go to player 1. (see Figure 1)

The next theorem states that the nucleolus (Schmeidler (1969), Kohlberg (1971)) of information games coincides with the τ-value (= the center of the core). We give a proof of this result based on Kohlberg's criterion. First we recall Kohlberg's criterion:

A collection C of coalitions is called a balanced collection if there are positive numbers $\lambda_S > 0$ for each $S \in C$ such that

$$\sum_{S\in C} \lambda_S 1_S = 1_N.$$

where $1_S \in \mathbb{R}^N$ is the vector with coordinates $(1_S)_i = 1$ if $i \in S$ and $(1_S)_i = 0$ if $i \notin S$. For every imputation $y \in A(v)$ of a cooperative game v we define

$$B_0 := \{S \subset N \mid |S| = 1 \text{ and } y(S) = v(S)\}$$

and for $k \geq 1$

$$B_k : = \{S \subset N \mid e(y,S) = \max \{e(y,T) \mid T \notin \bigcup_{j<k} B_j \}\}$$

where $e(y,S) = v(S) - y(S)$ is the excess of y in coalition S.

Now Kohlberg (1971) proved that y is the nucleolus iff for every $k \geq 1$ $\bigcup_{j \leq k} B_j$ contains a balanced collection C such that $\bigcup_{1 \leq j \leq k} B_j \subset C$

Lemma 4.2: If v an information game, then $y = \tau(v)$ satisfies Kohlbergs' criterion.

Proof: It is sufficient to prove that for every $p \in \mathbb{N}$ and every coalition $S \in B_p$, there exists a balanced collection $C \subset \bigcup_{q \leq p} B_p$ such that $S \in C$. This will follow from the following two assertions.

(i) For every coalition S, containing 1, we have:
$e(y,\{j\}) \geq e(y,S)$ for all $j \in N\backslash S$.

(ii) For every coalition S, not containing 1, we have:
$e(y,N\backslash\{i\}) \geq e(y,S)$ for all $i \in S$.

These two assertions are sufficient to prove the lemma.
For, if $S \in B_p$ and $1 \in S$ then

$C := \{S, \{j\}_{j \in N\backslash S}\}$ is a balanced collection and

$C \subset \bigcup_{q \leq p} B_q$ by (i).

If $S \in B_p$ and $1 \notin S$ then $C = \{S, \{N\backslash\{i\}\}_{i \in S}\}$ is balanced too and is contained in $\bigcup_{q \leq p} B_q$ by (ii).

To prove (i) an (ii) we compute some excessess.
Suppose $1 \in S$ and $j \notin S$. Then

$$e(y,S) = \sum_{T \cap S \neq \emptyset} r_T - (v(N) - \tfrac{1}{2}r(N\backslash\{1\})) - \tfrac{1}{2}r(S\backslash\{1\}) = - \sum_{T \cap S = \emptyset} r_T +$$

$$\tfrac{1}{2}r(N\backslash S) = - \sum_{T \cap S = \emptyset,\ |T| \geq 2} r_T - \tfrac{1}{2}r(N\backslash S) \leq - \tfrac{1}{2}r_j = e(y, \{j\}) \text{ proving (i).}$$

If $1 \notin S$ and $j \in S$ we find

$$e(y,S) = -\tfrac{1}{2}r(S) \leq -\tfrac{1}{2}r_j \text{ and}$$

$$e(y,N\backslash\{j\}) = v(N\backslash\{j\}) - y(N-\{j\}) = v(N\backslash\{j\}) - v(N) + y(N) - y(N-\{j\}) = -r_j + \tfrac{1}{2}r_j = -\tfrac{1}{2}r_j \text{ proving (ii).} \quad \square$$

Using Lemma 4.2 and Kohlberg's criterion we immediately have the following theorem.

Theorem 4.3: For each information game the τ-value is the nucleolus.

As a side result of the computation of the τ-value (theorem 4.1) and remark (ii) of section 2 we find:

Corollary 4.4: (Additivity of τ-value and nucleolus)
If v_1 and v_2 are information games, then

$$\tau[v_1 + v_2) = \tau(v_1) + \tau(v_2)$$

and the same holds for the nucleolus.

Now we turn our attention to the Shapley value $\phi(v)$ (Shapley (1953)). First we recall some properties of this solution concept:

(a) ϕ is linear on G^N
(b) $\phi(v) = \phi(v^*)$ for all $v \in G^N$ where v^* is the dual game of v
(c) $\phi(u_T) = |T|^{-1}\, 1_T$ where u_T is the simple game with one minimal winning coalition T.

Using these properties and theorem 3.1 we find for an information game

$$v = ru_1 + \sum_{T \subset N\backslash\{1\},\, T \neq \emptyset} r_T\, u^*_{T,1}$$

$$\phi(v) = r\,\phi(u_1) + \sum_{T:T \subset N\backslash\{1\}} r_T\, \phi(u^*_{T,1}) =$$

$$= r\, 1_{\{1\}} + \sum_{T:T \subset N\backslash\{1\}} r_T\, \phi(u_{T,1}).$$

Since $u_{T,1} = u_T + u_1 - u_{T\cup\{1\}}$ we find by linearity of ϕ

$$\phi(v) = r1_{\{1\}} + \sum_{T:T\subset N\setminus\{1\},T\neq\emptyset} r_T\{\frac{1}{|T|}1_T + 1_{\{1\}} - \frac{1}{|T|+1}1_{T\cup\{1\}}\}$$

$$= r1_{\{1\}} + \sum_{T:T\subset N\setminus\{1\},T\neq\emptyset} r_T\{\frac{|T|}{|T|+1}1_{\{1\}} + \frac{1}{|T|(|T|+1)}1_T\}.$$

Some Remarks

(i) From this formula for the Shapley value we may conclude that the Shapley value is more friendly for uninformed players than the nucleolus and the τ-value. For, from markets M_T, $T \subset N\setminus\{1\}$ and $|T| \geq 2$ the positive amount

$$r_T \frac{1}{|T|}\frac{1}{|T|+1}$$

goes to each of the players of T. Nucleolus and τ-value attribute the whole profit to the informed player 1.
The profit from markets M_T with $|T| = 1$ or $1 \in T$ is divided in the same way by the Shapley value, the nucleolus and the τ-value.

(ii) The Shapley value coincides with the τ-value iff

$$r_T = 0 \text{ for all } T \subset N\setminus\{1\} \text{ with } |T| \geq 2.$$

Viewing theorem 3.4, this is equivalent to convexity of the information game.

(iii) The Shapley value is, in general, not in the core e.g. if $N = \{1,2,3\}$ and $v = u^*_{\{2,3\},1}$ then

$$C(v) = \{(1,0,0)\} \text{ and } \phi(v) = (2/3,1/6,1/6).$$

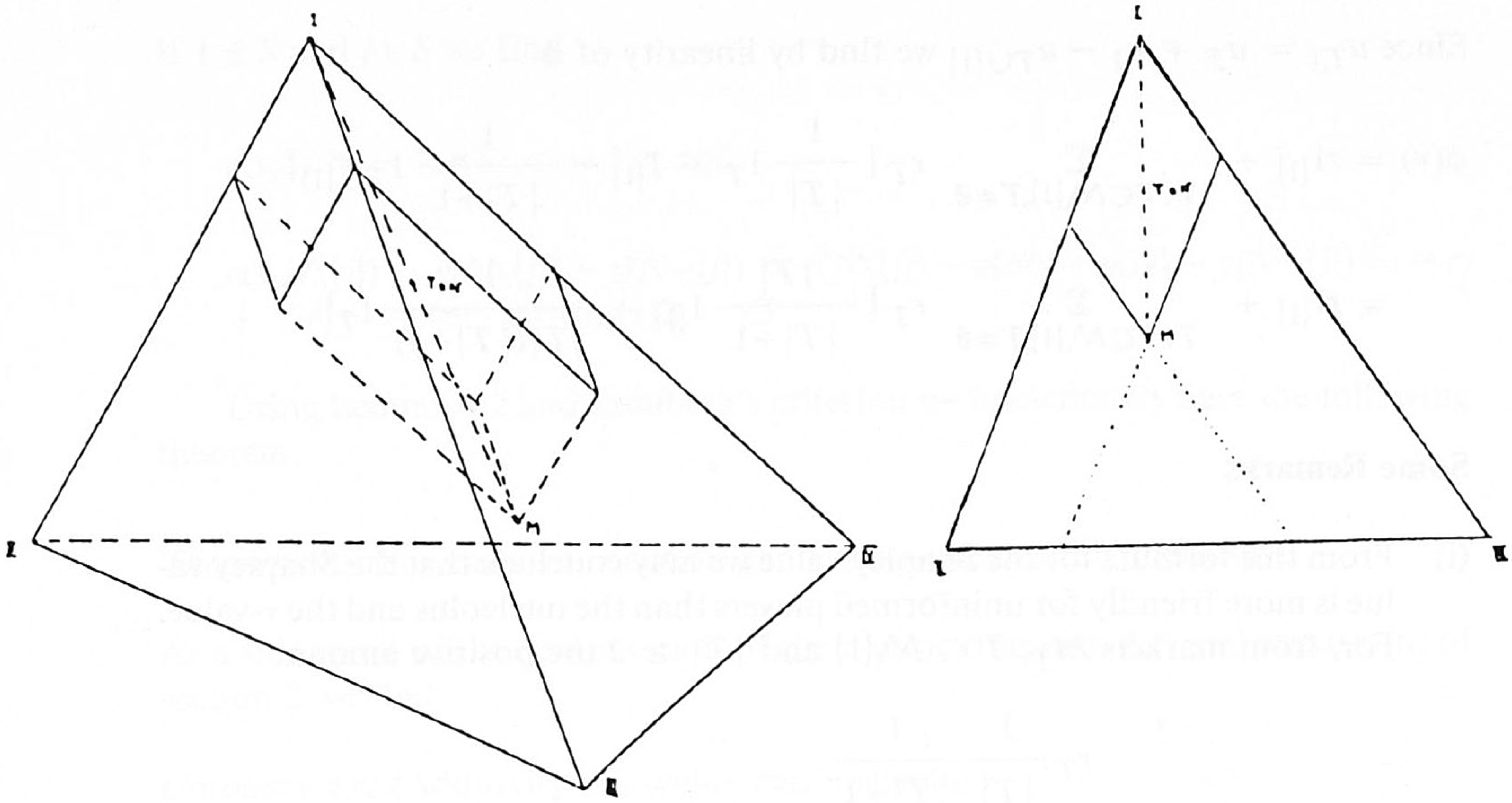

Fig. 1. The core of information market games with 3 and 4 players. The simplices I II III (IV) are the sets of imputations. The parallelogram and parallellopiped between the points I and M are the cores. The points $T = N$ are the coinciding τ-value and nucleolus. The points marked M are the imputations which give each non-informed player his marginal contribution M_i. If this point lies in the facet II III (IV), opposite to point I, then the game is convex.

6 Concluding Remarks

In this paper we examined a model of an information market and introduced a new type of cooperative games generated by information markets. The games obtained in this way form a cone of dimension 2^{n-1} (in the linear space of cooperative games G^N) generated by the simple games $u^*_{T,1}$ $(T \subset N\backslash\{1\},\ T \neq \emptyset)$ and u_1.

The cores of information games are hyperparallellopipeds and also minimal subsolutions of the game. The nucleolus and the τ-value of information games coincide with the center of the core and are additive on the cone of information games. The Shapley value coincides with the nucleolus and τ-value if and only the game is convex.

In Muto/Potters/Tijs (1987 a) the stable sets of information games are discussed in case that the uninformed players have equal possibilities in the game

i.e. r_T, $T \subset N\backslash\{1\}$ is only dependent on the cardinality of T.

Potters/Muto/Tijs (1988) examines the kernel and bargaining set of information games.

Potters/Tijs (1989) extends the result of this paper to a subclass of information markets with more than one initially informed players.

References

Gallini N T (1984) Deterrence by Market Sharing: A Strategic Incentive for Licensing. Amer Econ Rev 74: 931-941

Gallini N T, Winter R A (1984) Licensing in the Theory of Innovation. Working Paper no. 8419. Department of Economics and Institute for Policy Analysis. University of Toronto

Güth W (1984) How to Sell Valuable Information. Staatswissenschaftliches Seminar der Universität zu Köln. mimeo

Kamien M I, Tauman Y (1984) The Private Value of a Patent: A Game Theoretic Analysis. Zeitschrift für National Ökonomic suppl 4: 93-118

Kamien M I, Tauman Y (1986) Fees versus Royalties and the Private Value of a Patent. Quart J Econ CI: 471-491

Kamien M I, Tauman Y, Zang I (1985 b) Optimal License Fees for a New Product. Discussion Paper. Northwestern University

Kats A, Tauman Y (1982) Coalition Production Economies with Divisible and Indivisible Inputs (Asymptotic Results) J Math Ec 14: 19-42

Katz M L, Shapiro C (1984 a) On the licensing of Innovations. Discussion Paper no. 82. Woodrow Wilson School. Princeton University

Katz M L (1984 b) How to License a Patent. Discussion Paper no. 84. Woodrow Wilson School. Princeton University

Katz M L (1984 c) Perfect Equilibirum in a Development Game with Licensing or Immitation. Discussion Paper no. 85. Woodrow Wilson School. Princeton University

Kohlberg E (1971) On the Nucleolus of a Characteristic Function Game. SIAM J Appl Math 20: 62-65

Muto S (1986 a) An Information Good Market with Symmetric Externalities. Econometrica 54: 295-312

Muto S (1987) Possibility of Relicensing and Patent Protection. Euro Econ Rev 31: 927-945

Muto S, Potters J, Tijs S (1987 a) Stable Sets of Four-Person Symmetric Information Games. Discussion Paper 4. Tohoku Management and Accounting Research Group. Faculty of Economics. Tohoku University, Japan

Nakayama M (1987) Bargaining for an Information Good with Externalities. Discussion Paper no. 79. Faculty of Economics. Toyama University. Toyama. Japan

Neumann J von, Morgenstern O (1953) Theory of Games and Economic behaviour. Third Edition. Princeton University Press

Potters J, Muto S, Tijs S (1988) Bargaining Set and Kernel of Big Boss Games. Internal Report 88NICI10. Nijmeegs Institut voor Cognitieonderzoek en Informatie-technologie. Catholic University Nijmegen. The Netherlands.

Potters J, Tijs S (1989) Information Market Games with move then one informed firms Report 8910. Department of Mathematics. Catholic University Nijmegen. The Netherlands.

Roth A E (1976) Subsolutions and the Supercore of Cooperative Games. Math O R 1: 43-49

Schmeidler D (1969) The Nucleolus of a Characteristic Function Game. SIAM J Appl Math 17: 1163-1170

Shapley L S (1953) A Value for N-person Games. Ann Math Studies 28: 307-317

Shapley L S (1971) Cores of Convex Games. Int J G Th 1: 11-26

Tijs S H (1981) Bounds for the Core and the τ-value. In: Game Theory and Mathematical Economics (Eds. O. Moeschlin and D. Palaschke). North. Holland Publ. Cie 123-132

Tijs S H (1986) An axiomatization of the τ-value. Report 8603. Dept. of Maths, Catholic University Nijmegen. The Netherlands. (Math. Soc. Sci. 1986, forthcoming)

Received August 1986
Revised Version December 1987